Phase space Feynman path integrals of higher order parabolic type (Introductory Workshop on Path Integrals and Pseudo-Differential Operators)

熊ノ郷, 直人

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By

Naoto KUMANO-GO*

Abstract

This is a rough survey based on the talk at RIMS about the joint work [14] with A. S. Vasudeva Murthy (TIFR-CAM). In [14], we proved the existence of the phase space path integrals of higher order parabolic type with general functional as integrand. In this survey, we explain the process of its proof along the talk at RIMS, using some figures of paths.

§ 1. Introduction to Phase Space Path Integral

Let $0 < T \leq \mathbf{T} < \infty$, $x \in \mathbf{R}^n$ and $m > 0$. Let $U(T,0)$ be the fundamental solution for the m-th order parabolic equation such that

$$\Big(\partial_T + H(T,x,D_x)\Big)U(T,0) = O\,, \quad U(0,0) = I\,, \tag{1.1}$$

where $D_x = -i\partial_x$, O is the zero operator and I is the identity operator. By the Fourier transform with respect to $x_0 \in \mathbf{R}^n$ and the inverse Fourier transform with respect to $\xi_0 \in \mathbf{R}^n$, we can write

$$Iv(x) \equiv v(x) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbf{R}^{2n}} e^{i(x-x_0)\cdot\xi_0} v(x_0)dx_0 d\xi_0\,,$$

$$D_x v(x) = -i\partial_x v(x) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbf{R}^{2n}} e^{i(x-x_0)\cdot\xi_0} \xi_0 v(x_0)dx_0 d\xi_0\,,$$

$$H(T,x,D_x)v(x) \equiv \left(\frac{1}{2\pi}\right)^n \int_{\mathbf{R}^{2n}} e^{i(x-x_0)\cdot\xi_0} H(T,x,\xi_0)v(x_0)dx_0 d\xi_0\,.$$

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*Division of Liberal Arts, Kogakuin University, 2665-1, Nakanomachi, Hachiojishi, Tokyo, 192-0015, Japan.

We consider the symbol function $U(T,0,x,\xi_0)$ of $U(T,0)$ satisfying

$$(1.2)\qquad U(T,0)v(x)=\left(\frac{1}{2\pi}\right)^n\int_{\mathbf{R}^{2n}}e^{i(x-x_0)\cdot\xi_0}U(T,0,x,\xi_0)v(x_0)dx_0d\xi_0\,.$$

As an approximation of $U(T,0)$ as $T\downarrow 0$, we use the operator $I(T,0)$ defined by

$$I(T,0)v(x)\equiv\left(\frac{1}{2\pi}\right)^n\int_{\mathbf{R}^{2n}}e^{i(x-x_0)\cdot\xi_0}e^{-\int_0^T H(t,x,\xi_0)dt}v(x_0)dx_0d\xi_0\,.$$

Let $\Delta_{T,0}:T=T_{J+1}>T_J>\cdots>T_1>T_0=0$ be any division of the interval $[0,T]$ into subintervals. Note that $U(T,0)$ is a propagator. Then we have

$$U(T,0)v(x)=U(T,T_J)U(T_J,T_{J-1})\cdots U(T_2,T_1)U(T_1,0)v(x)\,.$$

Set $t_j=T_j-T_{j-1}$ and $|\Delta_{T,0}|=\max_{1\le j\le J+1}t_j$. When $|\Delta_{T,0}|\to 0$, we formally use $I(T_j,T_{j-1})$ as an approximation of $U(T_j,T_{j-1})$ and write

$$\begin{aligned}(1.3)\quad U(T,0)v(x)&=\lim_{|\Delta_{T,0}|\to 0}I(T,T_J)I(T_J,T_{J-1})\cdots I(T_2,T_1)I(T_1,0)v(x)\\&=\lim_{|\Delta_{T,0}|\to 0}\left(\frac{1}{2\pi}\right)^{n(J+1)}\int_{\mathbf{R}^{2n(J+1)}}e^{\sum_{j=1}^{J+1}(i(x_j-x_{j-1})\cdot\xi_{j-1}-\int_{T_{j-1}}^{T_j}H(t,x_j,\xi_{j-1})dt)}\\&\qquad\times v(x_0)\prod_{j=0}^{J}dx_jd\xi_j\end{aligned}$$

with $x=x_{J+1}$ (see [11, p.62, Remark 2°]). By (1.2) and (1.3), we can formally write

$$\begin{aligned}(1.4)\quad &e^{i(x-x_0)\cdot\xi_0}U(T,0,x,\xi_0)\\&=\lim_{|\Delta_{T,0}|\to 0}\left(\frac{1}{2\pi}\right)^{nJ}\int_{\mathbf{R}^{2nJ}}e^{\sum_{j=1}^{J+1}(i(x_j-x_{j-1})\cdot\xi_{j-1}-\int_{T_{j-1}}^{T_j}H(t,x_j,\xi_{j-1})dt)}\prod_{j=1}^{J}dx_jd\xi_j\,.\end{aligned}$$

According Feynman [5, Appendix B], we formally introduce a position path $X(t)$ with $X(T_j)=x_j$ and a momentum path $\Xi(t)$ with $\Xi(T_j)=\xi_j$ (though the author does not define the shapes of these paths in this stage, imagine Figure 1 for example). Then we can formally rewrite (1.4) as

$$(1.5)\qquad e^{i(x-x_0)\cdot\xi_0}U(T,0,x,\xi_0)=\int e^{i\phi(X,\Xi)}\mathcal{D}(X,\Xi)\,.$$

Here

$$(1.6)\qquad \phi(X,\Xi)\equiv\int_{[0,T)}\Xi(t)\cdot dX(t)+i\int_{[0,T)}H(t,X(t),\Xi(t))dt\,,$$

is the action for the paths (X,Ξ) on the phase space with $X(T)=x$ and $X(0)=x_0$ and $\Xi(0)=\xi_0$, and the phase space path integral $\int\sim\mathcal{D}(X,\Xi)$ is a sum over all the

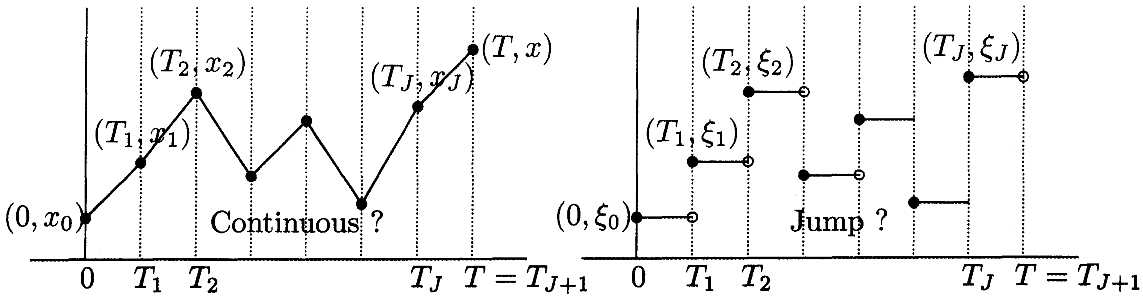

Figure 1. What is the position path X ? What is the momentum path Ξ ?

paths (X, Ξ). The expression on the right-hand side of (1.4) or (1.3) is now called the time slicing approximation for the path integral on the right-hand side of (1.5).

However, in the sense of mathematics, the measure $\mathcal{D}(X, \Xi)$ for the path integral of (1.5) which weighs all the paths (X, Ξ) equally, does not exist. Why can we say the path integral of (1.5) is a kind of integral ? Furthermore, in the sense of quantum mechanics, by the uncertainty principle, we can not have the position $X(t)$ and the momentum $\Xi(t)$ at the same time t. In (1.3), it seems that the approach via L^2-operator does not distinguish the difference between the configuration path integral and the phase space path integral. Why can we say the paths (X, Ξ) are phase space paths ? Furthermore, L. S. Schulman [16, p.303] says about phase space path integral that 'in this method formal trick of great power can give just plain wrong answers'.

In [14], using the time slicing approximation via piecewise constant paths, we proved the existence of the phase space Feynman path integrals

$$\int e^{i\phi(X,\Xi)} F(X,\Xi)\mathcal{D}(X,\Xi) \tag{1.7}$$

of the m-th order parabolic type with general functionals $F(X, \Xi)$. We regard (1.5) as the particular case of (1.7) with $F(X, \Xi) \equiv 1$. More precisely, we give a general class $\mathcal{F}$ of functionals $F(X, \Xi)$ so that for any $F(X, \Xi) \in \mathcal{F}$, the time slicing approximation of the phase space path integral (1.7) converges uniformly on compact subsets with the endpoint x of position paths X and the starting point ξ_0 of momentum paths Ξ (therefore, the author says that the paths (X, Ξ) are phase space paths). Furthermore, we proved some properties of the path integrals similar to the properties of the standard integrals. More precisely, though we need to pay attention for use, we proved the interchange of the order of the phase space path integrals with some integrals and some limits (therefore, the author says that the phase space path integral is a kind of integral).

Remark. For the phase space path integral of the Schrödinger type, there have been developed various mathematical approaches (cf. Schulman [16, §31]). For example, Daubechies-Klauder [4] via analytic continuation from measure, Albeverio-Guatteri-Mazzucchi [2][1, §10.5.3][15, §3.3] via Fresnel integral transform, Smolyanov-Tokarev-Truman [19] via Chernoff formula, Bock-Grothaus [3] via white noise analysis, H. Kumano-go-Kitada [8], N. Kumano-go [10], Ichinose[7] via Fourier integral operators and so on. Recently, N. Kumano-go-Fujiwara [12][13] proved the existence of the phase space path integrals of the Schrödinger type with general functional as integrand and proved their properties similar to some properties of standard integrals. On the other hand, in [14], we discussed the case of higher order parabolic type.

§ 2. Phase Space Path Integrals Exist

§ 2.1. Assumption for the symbol function $H(t,x,\xi)$

For the higher order parabolic equations which were discussed in C. Tsutsumi [17] and H. Kumano-go [9, §4 of Chapter 7], as we considered the phase space path integrals of (1.5) in [11], we consider the phase space path integrals of (1.7) with general functional $F(X,\Xi)$. In order to state the assumption for the symbol function $H(t,x,\xi)$ of (1.1), we need some notations.

Assumption 1. Let $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$. We say that a real-valued C^∞-function $\lambda(\xi)$ is a weight function if $\lambda(\xi)$ satisfies the following conditions:

(1) There exists a positive constant C_0 such that

$$1\leq\lambda(\xi)\leq C_0\langle\xi\rangle\,. \tag{2.1}$$

(2) For any multi-index β, there exists a positive constant C_β such that

$$\left|\partial_\xi^\beta\lambda(\xi)\right|\leq C_\beta\lambda(\xi)^{1-|\beta|}\,. \tag{2.2}$$

Example 2.1.

(1) $\lambda(\xi)=\langle\xi\rangle$.

(2) $\lambda(\xi)=(1+\sum_{k=1}^n|\xi^k|^{2m_k})^{1/(2m)}$ where $\xi=(\xi^1,\ldots,\xi^n)\in\mathbf{R}^n$, $m_k\in\mathbf{N}$, $k=1,2,\ldots,n$ and $m=\max_{1\leq k\leq n}m_k$.

Remark. Though the author use $\lambda(\xi)$ for generality, the reader may regard $\lambda(\xi)=\langle\xi\rangle$ for simplicity.

Remark. C. Tsutsumi [18] constructed the fundamental solution of (1.1) under more general conditions with a more general weight function $\lambda(x,\xi)$ depending on both x and ξ. However, we are yet to consider the path integrals under these conditions.

Our assumption for the symbol function $H(t,x,\xi)$ of (1.1) is the following (see also H. Kumano-go [9, Theorem 4.1 of Chapter 7]).

Assumption 2. Let $m>0$ and $0\le\delta<\rho\le 1$. Let $H(t,x,\xi)$ be a complex-valued C^∞-function satisfying the following conditions:

(1) There exist positive constants c, C such that

$$0<c\le \mathrm{Re}\, H(t,x,\xi)\le C\lambda(\xi)^m\,. \tag{2.3}$$

Here Re $H(t,x,\xi)$ is the real part of $H(t,x,\xi)$.

(2) For any multi-indices α, β, there exists a positive constant $C_{\alpha,\beta}$ such that

$$\left|D_x^\alpha\partial_\xi^\beta H(t,x,\xi)/\mathrm{Re}\, H(t,x,\xi)\right|\le C_{\alpha,\beta}\lambda(\xi)^{\delta|\alpha|-\rho|\beta|}\,. \tag{2.4}$$

Remark. We can treat the typical case of m-th order parabolic operator below.

(1) There exist positive constants c, C such that

$$0<c\lambda(\xi)^m\le \mathrm{Re}\, H(t,x,\xi)\le C\lambda(\xi)^m\,. \tag{2.5}$$

(2) For any multi-indices α, β, there exists a positive constant $C_{\alpha,\beta}$ such that

$$\left|D_x^\alpha\partial_\xi^\beta H(t,x,\xi)\right|\le C_{\alpha,\beta}\lambda(\xi)^{m+\delta|\alpha|-\rho|\beta|}\,. \tag{2.6}$$

Remark. Using the phase space path integrals with general functionals $F(X,\Xi)$ as integrand, we can construct the fundamental solution of a little more general parabolic operator $\partial_T+H'(T,x,D_x)$ (see Example 3.2).

Remark. Though the author use $0\le\delta<\rho\le 1$ for generality, the reader may regard $0=\delta<\rho=1$ for simplicity.

§ 2.2. Piecewise constant paths

Let $\Delta_{T,0}=(T,T_J,\dots,T_1,0)$ be any division of the interval $[0,T]$ given by

$$\Delta_{T,0}: T=T_{J+1}>T_J>\cdots>T_1>T_0=0\,. \tag{2.7}$$

Set $t_j=T_j-T_{j-1}$ and $|\Delta_{T,0}|=\max_{1\le j\le J+1}t_j$. Set $x_{J+1}=x$. Let $x_j\in\mathbf{R}^n$ and $\xi_j\in\mathbf{R}^n$. We define the position path $X_{\Delta_{T,0}}(t)=X_{\Delta_{T,0}}(t,x_{J+1},x_J,\dots,x_1,x_0)$ by

$$X_{\Delta_{T,0}}(0)=x_0\,,\quad X_{\Delta_{T,0}}(t)=x_j\,,\quad T_{j-1}<t\le T_j\,, \tag{2.8}$$

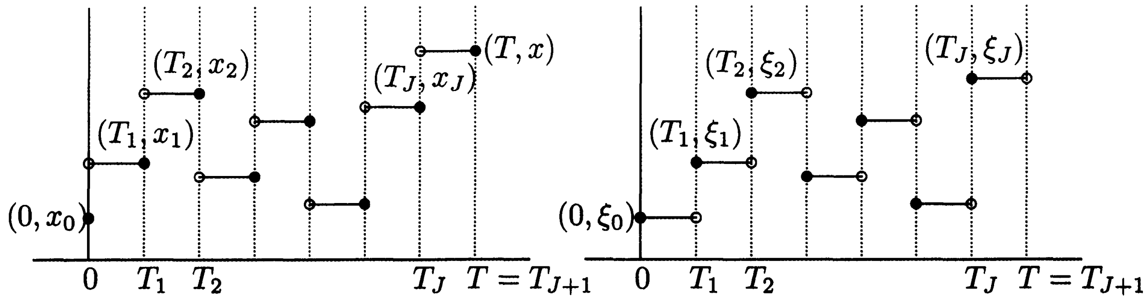

Figure 2. The position path $X_{\Delta_{T,0}}$ and the momentum path $\Xi_{\Delta_{T,0}}$

and the momentum path $\Xi_{\Delta_{T,0}}(t) = \Xi_{\Delta_{T,0}}(t, \xi_J, \ldots, \xi_1, \xi_0)$ by

$$\Xi_{\Delta_{T,0}}(t) = \xi_{j-1}\,, \quad T_{j-1} \le t < T_j \tag{2.9}$$

(see Figure 2).

Remark. $X_{\Delta_{T,0}}(t)$ is piecewise constant and left-continuous, and $\Xi_{\Delta_{T,0}}(t)$ is piecewise constant and right-continuous.

Then $\phi(X_{\Delta_{T,0}}, \Xi_{\Delta_{T,0}})$, $F(X_{\Delta_{T,0}}, \Xi_{\Delta_{T,0}})$ are the function $\phi_{\Delta_{T,0}}$, $F_{\Delta_{T,0}}$ given by

$$\begin{aligned} \phi(X_{\Delta_{T,0}}\Xi_{\Delta_{T,0}}) &\equiv \phi_{\Delta_{T,0}}(x_{J+1}, \xi_J, x_J, \ldots, \xi_1, x_1, \xi_0, x_0) \\ &= \sum_{j=1}^{J+1} \int_{[T_{j-1}, T_j)} \Xi_{\Delta_{T,0}} \cdot dX_{\Delta_{T,0}}(t) + i \sum_{j=1}^{J+1} \int_{[T_{j-1}, T_j)} H(t, X_{\Delta_{T,0}}, \Xi_{\Delta_{T,0}}) dt \\ &= \sum_{j=1}^{J+1} (x_j - x_{j-1}) \cdot \xi_{j-1} + i \sum_{j=1}^{J+1} \int_{T_{j-1}}^{T_j} H(t, x_j, \xi_{j-1}) dt\,, \end{aligned} \tag{2.10}$$

$$F(X_{\Delta_{T,0}}, \Xi_{\Delta_{T,0}}) \equiv F_{\Delta_{T,0}}(x_{J+1}, \xi_J, x_J, \ldots, \xi_1, x_1, \xi_0, x_0)\,. \tag{2.11}$$

§ 2.3. Time slicing approximation

Theorem 1 (Existence of phase space path integrals). *For any $F(X, \Xi) \in \mathcal{F}$, the time slicing approximation*

$$\begin{aligned} &\int e^{i\phi(X,\Xi)} F(X, \Xi) \mathcal{D}(X, \Xi) \\ &\equiv \lim_{|\Delta_{T,0}| \to 0} \left(\frac{1}{2\pi}\right)^{nJ} \int_{\mathbf{R}^{2nJ}} e^{i\phi(X_{\Delta_{T,0}}, \Xi_{\Delta_{T,0}})} F(X_{\Delta_{T,0}}, \Xi_{\Delta_{T,0}}) \prod_{j=1}^{J} d\xi_j dx_j\,, \end{aligned} \tag{2.12}$$

converges uniformly on compact sets of (x, ξ_0, x_0), i.e., the phase space path integral is well defined.

Remark. Even when $F(X,\Xi) \equiv 1$, each integral of the right hand side

$$\lim_{|\Delta_{T,0}|\to 0} \left(\frac{1}{2\pi}\right)^{nJ} \int_{\mathbf{R}^{2nJ}} e^{i\sum_{j=1}^{J+1}(x_j-x_{j-1})\cdot\xi_{j-1}-\sum_{j=1}^{J+1}\int_{T_{j-1}}^{T_j} H(t,x_j,\xi_{j-1})dt} \prod_{j=1}^{J} d\xi_j dx_j \,,$$

of (2.12) does not always converges absolutely, i.e., $\int_{\mathbf{R}^{2n}} dx_j d\xi_j = \infty$. Furthermore, the number J of integrals tends to ∞, i.e., $\infty \times \infty \times \infty \times \cdots\cdots$ as $J \to \infty$. Therefore we treat the multiple integral of (2.12) as an oscillatory integral (cf. H. Kumano-go [9, §1.6]).

In Theorem 1, the definition of the class $\mathcal{F}$ is essential. However, for the sake of simplicity, we will explain how to define the class $\mathcal{F}$ of functionals $F(X,\Xi)$ in the last section (for the definition of $\mathcal{F}$, see Definition 1 of §5).

§ 3. We Can Produce Many Functionals $F(X,\Xi) \in \mathcal{F}$.

§ 3.1. Algebra on the class $\mathcal{F}$

First we explain the property of the class $\mathcal{F}$ of functionals $F(X,\Xi)$.

Theorem 2 (Algebra on $\mathcal{F}$). *For any $F(X,\Xi) \in \mathcal{F}$ and $G(X,\Xi) \in \mathcal{F}$, we have*

$$F(X,\Xi) + G(X,\Xi) \in \mathcal{F}\,, \ F(X,\Xi)G(X,\Xi) \in \mathcal{F}\,. \tag{3.1}$$

Remark. In other words, $\mathcal{F}$ is closed under addition and multiplication. If we apply Theorem 2 to the examples of $F(X,\Xi) \in \mathcal{F}$ in Example 3.1, we can produce many functionals $F(X,\Xi) \in \mathcal{F}$ which are 'phase space path integrable'.

§ 3.2. Examples of $F(X,\Xi) \in \mathcal{F}$

Next we explain typical examples of functionals $F(X,\Xi)$ belonging to $\mathcal{F}$.

Example 3.1. Let $0 \le \delta < \rho \le 1$, $L \ge 0$ and $0 \le T' \le T'' \le T$.

(1) Assume that for any multi-index α, $D_x^\alpha B(t,x)$ is continuous and satisfies

$$|D_x^\alpha B(t,x)| \le C_\alpha \langle x\rangle^L \tag{3.2}$$

with a positive constant C_α. Then the value at a fixed time t $(0 \le t \le T)$,

$$F(X) \equiv B(t, X(t)) \in \mathcal{F}\,. \tag{3.3}$$

In particular, $F(X) \equiv 1 \in \mathcal{F}$ and $F(X) \equiv X(t) \in \mathcal{F}$.

(2) Assume that for any multi-indices α, β, $D_x^\alpha \partial_\xi^\beta B(t,x,\xi)$ is continuous and satisfies

$$\left|D_x^\alpha \partial_\xi^\beta B(t,x,\xi)\right| \le C_{\alpha,\beta} \left(\langle x\rangle + \lambda(\xi)\right)^L \lambda(\xi)^{\delta|\alpha|-\rho|\delta|} \tag{3.4}$$

with a positive constant $C_{\alpha,\beta}$. Then we have

$$F(X,\Xi) \equiv \int_{[T',T'')} B(t,X(t),\Xi(t))dt \in \mathcal{F}. \tag{3.5}$$

(3) Assume that for any multi-indices α, β, $D_x^\alpha \partial_\xi^\beta B(t,x,\xi)$ is continuous and satisfies

$$\left|D_x^\alpha \partial_\xi^\beta B(t,x,\xi)\right| \le C_{\alpha,\beta} \lambda(\xi)^{\delta|\alpha|-\rho|\delta|} \tag{3.6}$$

with a positive constant $C_{\alpha,\beta}$. Then we have

$$F(X,\Xi) \equiv e^{\int_{[T',T'')} B(t,X(t),\Xi(t))dt} \in \mathcal{F}. \tag{3.7}$$

(4) Assume that for any multi-index α, $D_x^\alpha B(t,x)$ is continuous and satisfies

$$0 < c \le \operatorname{Re} B(t,x) \le C\langle x\rangle^L, \quad |D_x^\alpha B(t,x)/\operatorname{Re} B(t,x)| \le C_\alpha$$

with some positive constants c, C, C_α. Then we have

$$F(X) \equiv e^{-\int_{[T',T'')} B(t,X(t))dt} \in \mathcal{F}.$$

Example 3.2. Assume that for any multi-index α, $D_x^\alpha a(t,x)$ is continuous and satisfies $0 < c' < a(t,x)$ and $|D_x^\alpha a(t,x)| \le C'_\alpha$ with some positive constants c', C'_α. We consider the parabolic operator

$$\partial_T + H'(T,x,D_x) \equiv \partial_T + a(T,x)|D_x|^{2k} + |x|^{2l} \tag{3.8}$$

with non-negative integers k, l. Set $H(t,x,\xi) = a(t,x)|\xi|^{2k} + 1$, $B_1(x) = -2$ and $B_2(x) = |x|^{2l} + 1$. Then we have

$$H'(t,x,\xi) = H(t,x,\xi) + B_1(x) + B_2(x). \tag{3.9}$$

Let

$$\phi(X,\Xi) \equiv \int_{[0,T)} \Xi(t)\cdot dX(t) + i\int_{[0,T)} H(t,X(t),\Xi(t))dt.$$

By Example 3.1(3)(4), we get

$$F(X) \equiv e^{-\int_{[0,T)} B_1(X(t))dt} \in \mathcal{F}, \quad G(X) \equiv e^{-\int_{[0,T)} B_2(X(t))dt} \in \mathcal{F}. \tag{3.10}$$

By Theorem 2, we have $F(X)G(X) \in \mathcal{F}$. Therefore we can write the symbol function $U'(T,0,x,\xi_0)$ of the fundamental solution $U'(T,0)$ for the parabolic equation such that

$$\Big(\partial_T + H'(T,x,D_x)\Big)U'(T,0) = O\,, \quad U'(0,0) = I \tag{3.11}$$

in the path integral form

$$\begin{aligned} e^{i(x-x_0)\cdot\xi_0}U'(T,0,x,\xi_0) &= \int e^{i\int_{[0,T)}\Xi(t)\cdot dX(t) - \int_{[0,T)} H'(t,X(t),\Xi(t))dt}\mathcal{D}(X,\Xi) \\ &\equiv \int e^{i\phi(X,\Xi)}F(X)G(X)\mathcal{D}(X,\Xi)\,. \end{aligned} \tag{3.12}$$

§ 4. Theorem of Fubini Type

Though the measure $\mathcal{D}(X,\Xi)$ of the phase space path integral (2.12) does not exist, we can interchange the order of the phase space path integration and the integration with respect to time.

Theorem 3 (Interchange of the order with the integral with respect to time). *Let $L \geq 0$ and $0 \leq T' \leq T'' \leq T$. Assume that for any multi-index α, $D_x^\alpha B(t,x)$ is continuous and satisfies*

$$|D_x^\alpha B(t,x)| \leq C_\alpha \langle x\rangle^L \tag{4.1}$$

with a positive constant C_α. Then, for any $F(X,\Xi) \in \mathcal{F}$ including $F(X,\Xi) \equiv 1$, we have

$$\begin{aligned} &\int e^{i\phi(X,\Xi)}\left(\int_{[T',T'')} B(t,X(t))dt\right)F(X,\Xi)\mathcal{D}(X,\Xi) \\ &= \int_{[T',T'')}\left(\int e^{i\phi(X,\Xi)}B(t,X(t))F(X,\Xi)\mathcal{D}(X,\Xi)\right)dt\,. \end{aligned} \tag{4.2}$$

Remark. To avoid the uncertain principle, we do not treat $B(t,X(t),\Xi(t))$, i.e., the position $X(t)$ and the momentum $\Xi(t)$ at the same time t.

Remark. We can also interchange the order of the phase space path integration and some limits (for the details, see [14, Theorem 5]). For example, assume that for any multi-index α, $D_x^\alpha B(t,x)$ is continuous and satisfies $|D_x^\alpha B(t,x)| \leq C_\alpha$ with a positive constant C_α. Then we have the perturbation expansion formula

$$\begin{aligned} &\int e^{i\phi(X,\Xi)+\int_{[T',T'')}B(t,X(t))dt}\mathcal{D}(X,\Xi) \\ &= \sum_{k=0}^{\infty}\int_{[T',T'')}dt_k\int_{[T',t_k)}dt_{k-1}\cdots\int_{[T',t_2)}dt_1 \\ &\quad \times \int e^{i\phi(X,\Xi)}B(t_k,X(t_k))B(t_{k-1},X(t_{k-1}))\cdots B(t_1,X(t_1))\mathcal{D}(X,\Xi)\,. \end{aligned} \tag{4.3}$$

§ 5. How to Define The Class $\mathcal{F}$

§ 5.1. Process of proof for Theorems 1 and 2

We explain the process of the proof of Theorems 1 and 2 (for the details of the proof, see [14]).

In order to prove the existence of the phase space path integral (2.12), i.e., the convergence of the multiple oscillatory integral

$$
\begin{aligned}
(5.1) \qquad & \left(\frac{1}{2\pi}\right)^{nJ} \int_{\mathbf{R}^{2nJ}} e^{i\phi(X_{\Delta_{T,0}},\Xi_{\Delta_{T,0}})} F(X_{\Delta_{T,0}},\Xi_{\Delta_{T,0}}) \prod_{j=1}^{J} d\xi_j dx_j\,, \\
& \equiv e^{i(x_{J+1}-x_0)\cdot\xi_0} q_{\Delta_{T,0}}(x_{J+1},\xi_0,x_0)
\end{aligned}
$$

as $|\Delta_{T,0}| \to 0$, we have only to add many assumptions to

$$
(5.2) \qquad F(X_{\Delta_{T,0}},\Xi_{\Delta_{T,0}}) \equiv F_{\Delta_{T,0}}(x_{J+1},\xi_J,x_J,\ldots,\xi_1,x_1,\xi_0,x_0)\,,
$$

and to define the class $\mathcal{F}$ of functionals $F(X,\Xi)$ by these assumptions. Because we have not given any assumption to $F(X,\Xi)$ until this section, we need some assumption.

Do not consider other things. Then $\mathcal{F}$ will become larger as a set. If we are lucky, $\mathcal{F}$ will contain at least one example $F(X,\Xi) \equiv 1$.

The assumptions should be closed under addition and multiplication. Then $\mathcal{F}$ will also be closed under addition and multiplication, i.e., Theorem 2 will hold.

Our proof consists of 3 steps.

1° We control the multiple integral of (5.1) by C^J as $J \to \infty$ with a positive constant C (Estimate of H. Kumano-go-Taniguchi's type, cf. [9, Lemma 2.2 of Chapter 7]).

2° We control the multiple integral of (5.1) by a positive constant C independent of $J \to \infty$ (Estimate of Fujiwara's type, cf. [6]).

3° We add assumptions so that the multiple integral (5.1) converges as $|\Delta_{T,0}| \to 0$.

Remark. When $F_{\Delta_{T,0}}$ is independent of x_0 (for example, $F(X,\Xi) \equiv 1$), the symbol function $q_{\Delta_{T,0}}(x_{J+1},\xi_0,x_0)$ of (5.1) is also independent of x_0, i.e., we can write $q_{\Delta_{T,0}}(x_{J+1},\xi_0,x_0)$ as $q_{\Delta_{T,0}}(x_{J+1},\xi_0)$.

§ 5.2. Estimate of H. Kumano-go-Taniguchi's type

In order to control the multiple integral (5.1) by C^J as $J \to \infty$ with a positive constant C, we assume that we can control $F_{\Delta_{T,0}}$ by $(B_{\ell_1,\ell_2})^J$ as follows.

Tentative Assumption 1. Let $0 < T \leq \mathbf{T}$. Let A and L be non-negative constants. For any non-negative integers ℓ_1, ℓ_2, there exists a positive constant B_{ℓ_1,ℓ_2} such that for any division $\Delta_{T,0}$, any multi-indices α_j, β_{j-1} with $|\alpha_j| \le \ell_1$, $|\beta_{j-1}| \le \ell_2$, $j = 1, 2, \ldots, J, J+1$,

$$\left|\left(\prod_{j=1}^{J+1} D_{x_j}^{\alpha_j}\partial_{\xi_{j-1}}^{\beta_{j-1}}\right) F_{\Delta_{T,0}}(x_{J+1}, \xi_J, x_J, \ldots, \xi_1, x_1, \xi_0, x_0)\right| \tag{5.3}$$

$$\le A(B_{\ell_1,\ell_2})^{J+1}\left(\sum_{j=1}^{J+1}\langle x_j\rangle + \sum_{j=1}^{J+1}\lambda(\xi_{j-1}) + \langle x_0\rangle\right)^L \prod_{j=1}^{J+1}\lambda(\xi_{j-1})^{\delta|\alpha_j|-\rho|\beta_{j-1}|}.$$

Remark. All functionals $F(X,\Xi)$ of Example 3.1 satisfy Tentative Assumption 1: For example, we consider $F(X) \equiv B(t, X(t))$ with $|D_x^\alpha B(t,x)| \le C_\alpha \langle x\rangle^L$ of Example 3.1 (1). Using k such that $T_{k-1} < t \le T_k$, we can write

$$F_{\Delta_{T,0}}(x_k) = B(t, x_k). \tag{5.4}$$

Therefore we can show that $F_{\Delta_{T,0}}$ satisfies (5.3). Next we consider

$$F(X,\Xi) \equiv e^{\int_{[0,T)} B(t,X(t),\Xi(t))dt}$$

with $|D_x^\alpha \partial_\xi^\beta B(t,x,\xi)| \le C_{\alpha,\beta}\lambda(\xi)^{\delta|\alpha|-\rho|\beta|}$ of Example 3.1 (3) with $0 = T' < T'' = T$. We can write

$$F_{\Delta_{T,0}} = \prod_{j=1}^{J+1} e^{\int_{[T_{j-1},T_j)} B(t,x_j,\xi_{j-1})dt} \tag{5.5}$$

Therefore we can show that $F_{\Delta_{T,0}}$ satisfies (5.3).

Remark. Note that

$$e^{i\phi_{\Delta_{T,0}}} = e^{i\sum_{j=1}^{J+1}(x_j - x_{j-1})\cdot\xi_{j-1}} e^{-\sum_{j=1}^{J+1}\int_{[T_{j-1},T_j)} H(t,x_j,\xi_{j-1})dt}. \tag{5.6}$$

In a view of pseudo-differential operators with multiple-symbol of [9, §2 of Chapter 7], we treat

$$p \equiv p(x_{J+1}, \xi_J, x_J, \ldots, \xi_1, x_1, \xi_0, x_0) = e^{-\sum_{j=1}^{J+1}\int_{[T_{j-1},T_j)} H(t,x_j,\xi_{j-1})dt} F_{\Delta_{T,0}} \tag{5.7}$$

as a multiple symbol function.

Under Tentative Assumption 1, we can control $q_{\Delta_{T,0}}(x_{J+1}, \xi_0, x_0)$ of (5.1) by $(C_{\ell_1,\ell_2})^J$ as $J \to \infty$ with a positive constant C_{ℓ_1,ℓ_2} as follows.

Lemma 5.1. *Let* $0 < T \le \mathbf{T}$. *For any non-negative integers* ℓ_1, ℓ_2, *there exists a positive constant* C_{ℓ_1,ℓ_2} *such that*

$$\left|D_x^\alpha \partial_{\xi_0}^\beta q_{\Delta_{T,0}}(x, \xi_0, x_0)\right| \le A(C_{\ell_1,\ell_2})^J \left(\langle x\rangle + \lambda(\xi_0) + \langle x_0\rangle\right)^L \lambda(\xi_0)^{\delta|\alpha|-\rho|\beta|} \tag{5.8}$$

for any division $\Delta_{T,0}$ *and any multi-indices* α, β *with* $|\alpha| \le \ell_1$ *and* $|\beta| \le \ell_2$.

§ 5.3. Estimate of Fujiwara's type

In order to control the multiple integral (5.1) by a positive constant C independent of $J \to \infty$, we add the term $\prod_{j=1}^{J+1} (t_j)^{\min(|\beta_{j-1}|,1)}$ to Tentative Assumption 1 as follows.

Tentative Assumption 2. Let $0 < T \leq \mathbf{T}$. Let A and L be non-negative constants. For any non-negative integers ℓ_1, ℓ_2, there exists a positive constant B_{ℓ_1,ℓ_2} such that for any division $\Delta_{T,0}$, any multi-indices α_j, β_{j-1} with $|\alpha_j| \leq \ell_1$, $|\beta_{j-1}| \leq \ell_2$, $j = 1, 2, \dots, J, J+1$,

$$\left| \left(\prod_{j=1}^{J+1} D_{x_j}^{\alpha_j} \partial_{\xi_{j-1}}^{\beta_{j-1}} \right) F_{\Delta_{T,0}}(x_{J+1}, \xi_J, x_J, \dots, \xi_1, x_1, \xi_0, x_0) \right| \tag{5.9}$$

$$\leq A(B_{\ell_1,\ell_2})^{J+1} \prod_{j=1}^{J+1} (t_j)^{\min(|\beta_{j-1}|,1)}$$

$$\times \left(\sum_{j=1}^{J+1} \langle x_j \rangle + \sum_{j=1}^{J+1} \lambda(\xi_{j-1}) + \langle x_0 \rangle \right)^L \prod_{j=1}^{J+1} \lambda(\xi_{j-1})^{\delta|\alpha_j| - \rho|\beta_{j-1}|}.$$

Remark. All functionals $F(X, \Xi)$ of Example 3.1 satisfy Tentative Assumption 2: For example, we consider $F(X) \equiv B(t, X(t))$ with $|D_x^\alpha B(t,x)| \leq C_\alpha \langle x \rangle^L$ of Example 3.1 (1). Let $T_{k-1} < t \leq T_k$. If $|\beta_{j-1}| \geq 1$, then we have

$$\partial_{\xi_{j-1}}^{\beta_{j-1}} F_{\Delta_{T,0}} = \partial_{\xi_{j-1}}^{\beta_{j-1}} B(t, x_k) = 0 \leq t_j\,. \tag{5.10}$$

Therefore we can show that $F_{\Delta_{T,0}}$ satisfies (5.9). Next we consider

$$F(X, \Xi) \equiv e^{\int_{[0,T)} B(t, X(t), \Xi(t))dt}$$

with $|D_x^\alpha \partial_\xi^\beta B(t,x,\xi)| \leq C_{\alpha,\beta} \lambda(\xi)^{\delta|\alpha| - \rho|\beta|}$ of Example 3.1 (3) with $0 = T' < T'' = T$. For multi-index e with $|e| = 1$ and $1 \leq k \leq J+1$, we can write

$$\partial_{\xi_{k-1}}^e F_{\Delta_{T,0}} = \prod_{j=1}^{J+1} e^{\int_{[T_{j-1},T_j)} B(t, x_j, \xi_{j-1})dt} \times \int_{[T_{k-1},T_k)} (\partial_\xi^e B)(t, x_k, \xi_{k-1})dt\,. \tag{5.11}$$

Therefore we can show that $F_{\Delta_{T,0}}$ satisfies (5.9).

Remark. Using the figures of paths, we explain the cases of (5.9) with $J = 0, 1, 2$.

- If $J = 0$ (see the "one-piece" paths of Figure 3), $F_{T,0}(x_1, \xi_0, x_0)$ can be controlled by $(B_{\ell_1,\ell_2})^1$.
- If $J = 1$ (see the "two-piece" paths of Figure 4), $F_{T,T_1,0}(x_2, \xi_1, x_1, \xi_0, x_0)$ can be controlled by $(B_{\ell_1,\ell_2})^2$.

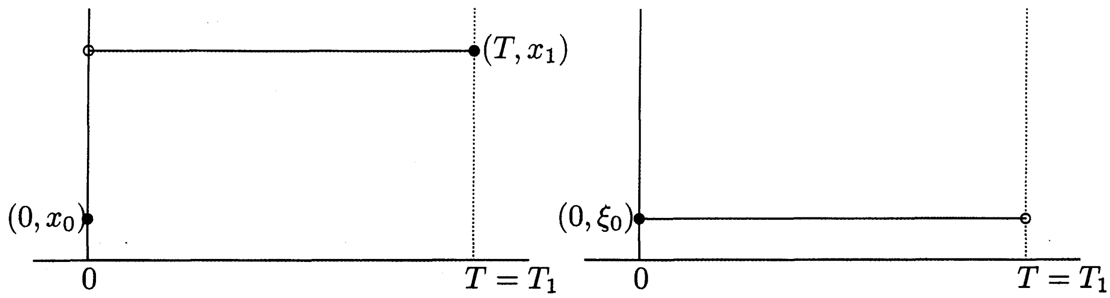

Figure 3. The "one-piece" paths $X_{T,0}(x_1, x_0)$ and $\Xi_{T,0}(\xi_0)$

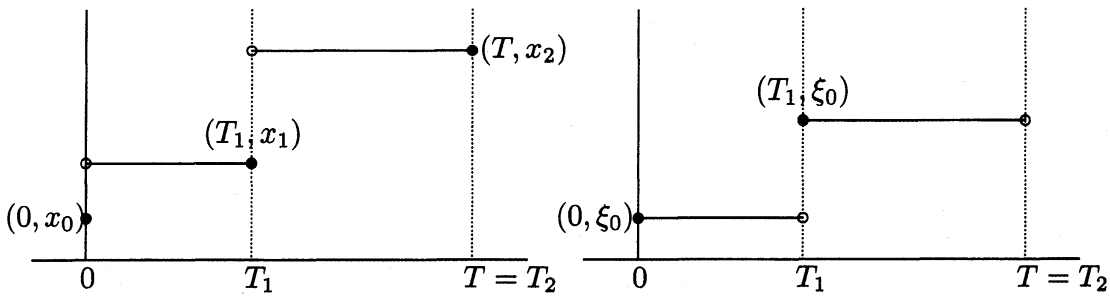

Figure 4. The "two-piece" paths $X_{T,T_1,0}(x_2, x_1, x_0)$ and $\Xi_{T,T_1,0}(\xi_1, \xi_0)$

- If $J = 2$ (see the "three-piece" paths of Figure 5), $F_{T,T_2,T_1,0}(x_3, \xi_2, x_2, \xi_1, x_1, \xi_0, x_0)$ can be controlled by $(B_{\ell_1,\ell_2})^3$.

Furthermore, we note the following lemma.

Lemma 5.2. *If* $x_1 = x_2$ *and* $\xi_1 = \xi_0$*, then we have*

$$F_{T_2,T_1,0}(x_2, \xi_0, x_2, \xi_0, x_0) = F_{T_2,0}(x_2, \xi_0, x_0)$$

Proof. When $x_1 = x_2$ and $\xi_1 = \xi_0$ (see Figure 6), we have

$$\begin{aligned} F_{T_2,T_1,0}(x_2, \xi_0, x_2, \xi_0, x_0) &= F(X_{T_2,T_1,0}(x_2, x_2, x_0), \Xi_{T_2,T_1,0}(\xi_0, \xi_0)) \\ &= F(X_{T_2,0}(x_2, x_0), \Xi_{T_2,0}(\xi_0)) = F_{T_2,0}(x_2, \xi_0, x_0)\,. \end{aligned}$$

□

Let N be a positive integer with $2m - (\rho - \delta)N \leq 0$. We repeating the asymptotic expansion formula

$$\sum_{|\alpha_1|<N} \frac{1}{\alpha!}(\partial_{\xi_1}^{\alpha_1} D_{x_1}^{\alpha_1} p)(x_2, \xi_0, x_2, \xi_0) + r_N(x_2, \xi_0) \tag{5.12}$$

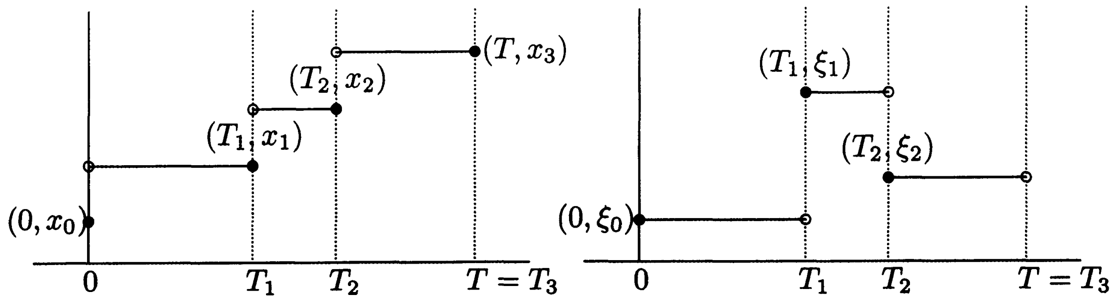

Figure 5. The "three-piece" paths $q_{T,T_2,T_1,0}(x_3, x_2, x_1, x_0)$ and $p_{T,T_2,T_1,0}(\xi_2, \xi_1, \xi_0)$

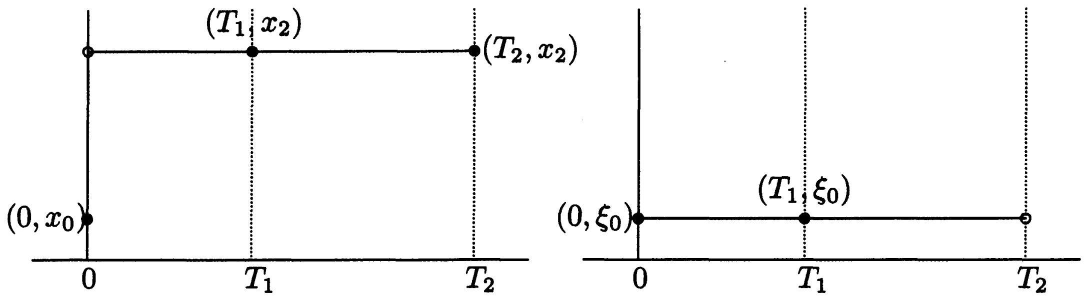

Figure 6. The paths $X_{T_2,T_1,0}(x_2, x_2, x_0)$ and $\Xi_{T_2,T_1,0}(\xi_0, \xi_0)$

for the pseudo-differential operator with the double symbol $p(x_2, \xi_1, x_1, \xi_0)$ (see [9, §3 of Chapter 2]), we change the paths of the main terms into simpler paths over and over again. Then the many remainder terms terms appear. However the many remainder terms can be controlled by the many terms t_j of (5.9). Using the many terms t_j, we control the sum of the many remainder terms with $\sum_{j=1}^{J+1} t_j = T$ and $Te^T \leq Te^{\mathbf{T}} \leq CT$.

Remark. For the pseudo-differential operators with the multiple-symbol

$$p = e^{-\sum_{j=1}^{J+1} \int_{[T_{j-1}, T_j)} H(t, x_j, \xi_{j-1}) dt} F_{\Delta_{T,0}}\,,$$

we treat

$$\sum_{\sum_{j=1}^{J} |\alpha_j| < N} \frac{1}{\prod_{j=1}^{J} \alpha_j!} \left(\partial_{\xi_J}^{\alpha_J} D_{x_J}^{\alpha_J} \cdots \left(\partial_{\xi_2}^{\alpha_2} D_{x_2}^{\alpha_2} \left(\partial_{\xi_1}^{\alpha_1} D_{x_1}^{\alpha_1} p \right) \Big|_{\xi_1 = \xi_0}^{x_1 = x_2} \right) \Big|_{\xi_2 = \xi_0}^{x_2 = x_3} \right) \cdots \Bigg) \Bigg|_{\xi_J = \xi_0}^{x_J = x_{J+1}}$$

as the main term of the asymptotic expansion.

Using $F_{T,0}(x, \xi_0, x_0) = F(X_{T,0}, \Xi_{T,0})$, we set

$$q_{T,0}(x, \xi_0, x_0) = e^{-\int_{[0,T)} H(t, x, \xi_0) dt} F_{T,0}(x, \xi_0, x_0)\,. \tag{5.13}$$

Under Tentative Assumption 2, we can control $q_{\Delta_{T,0}}(x_{J+1},\xi_0,x_0)$ of (5.1) by a positive constant C_{ℓ_1,ℓ_2} independent of $J\to\infty$ as follows.

Lemma 5.3. *Let $0<T\leq \mathbf{T}$. For any non-negative integers ℓ_1, ℓ_2, there exist positive constants C_{ℓ_1,ℓ_2}, C'_{ℓ_1,ℓ_2} such that*

$$\left|D_x^\alpha\partial_{\xi_0}^\beta q_{\Delta_{T,0}}(x,\xi_0,x_0)\right|\leq AC_{\ell_1,\ell_2}\left(\langle x\rangle+\lambda(\xi_0)+\langle x_0\rangle\right)^L\lambda(\xi_0)^{\delta|\alpha|-\rho|\beta|},\tag{5.14}$$

and

$$\begin{aligned}&\left|D_x^\alpha\partial_{\xi_0}^\beta\left(q_{\Delta_{T,0}}(x,\xi_0,x_0)-q_{T,0}(x,\xi_0,x_0)\right)\right|\\&\leq AC'_{\ell_1,\ell_2}T\left(\langle x\rangle+\lambda(\xi_0)+\langle x_0\rangle\right)^L\lambda(\xi_0)^{2m+\delta|\alpha|-\rho|\beta|}\end{aligned}\tag{5.15}$$

for any division $\Delta_{T,0}$ and any multi-indices α, β with $|\alpha|\leq\ell_1$ and $|\beta|\leq\ell_2$.

§ 5.4. The class $\mathcal{F}$ of functionals $F(X,\Xi)$

We add (5.17) to Tentative Assumption 2 as follows so that the multiple integral (5.1) converges as $|\Delta_{T,0}|\to\infty$.

Assumption 3. Let $0<T\leq\mathbf{T}$. Let A and L be non-negative constants. Let μ be a positive bounded Borel measure on $[0,T]$. For any non-negative integers ℓ_1, ℓ_2, there exists a positive constant B_{ℓ_1,ℓ_2} such that for any division $\Delta_{T,0}$, any multi-indices α_j, β_{j-1} with $|\alpha_j|\leq\ell_1$, $|\beta_{j-1}|\leq\ell_2$, $j=1,2,\ldots,J,J+1$,

$$\begin{aligned}&\left|\left(\prod_{j=1}^{J+1}D_{x_j}^{\alpha_j}\partial_{\xi_{j-1}}^{\beta_{j-1}}\right)F_{\Delta_{T,0}}(x_{J+1},\xi_J,x_J,\ldots,\xi_1,x_1,\xi_0,x_0)\right|\\&\leq A(B_{\ell_1,\ell_2})^{J+1}\prod_{j=1}^{J+1}(t_j)^{\min(|\beta_{j-1}|,1)}\\&\quad\times\left(\sum_{j=1}^{J+1}\langle x_j\rangle+\sum_{j=1}^{J+1}\lambda(\xi_{j-1})+\langle x_0\rangle\right)^L\prod_{j=1}^{J+1}\lambda(\xi_{j-1})^{\delta|\alpha_j|-\rho|\beta_{j-1}|},\end{aligned}\tag{5.16}$$

and, for any integer s with $1\leq s\leq J+1$, if $|\alpha_s|>0$,

$$\begin{aligned}&\left|\left(\prod_{j=1}^{J+1}D_{x_j}^{\alpha_j}\partial_{\xi_{j-1}}^{\beta_{j-1}}\right)F_{\Delta_{T,0}}(x_{J+1},\xi_J,x_J,\ldots,\xi_1,x_1,\xi_0,x_0)\right|\\&\leq A(B_{\ell_1,\ell_2})^{J+1}\mu((T_{s-1},T_s])\prod_{j=1,\,j\neq s}^{J+1}(t_j)^{\min(|\beta_{j-1}|,1)}\\&\quad\times\left(\sum_{j=1}^{J+1}\langle x_j\rangle+\sum_{j=1}^{J+1}\lambda(\xi_{j-1})+\langle x_0\rangle\right)^L\prod_{j=1}^{J+1}\lambda(\xi_{j-1})^{\delta|\alpha_j|-\rho|\beta_{j-1}|},\end{aligned}\tag{5.17}$$

where $\mu((T_{s-1},T_s])$ is the measure μ of the interval $(T_{s-1},T_s]$.

Remark. All functionals $F(X,\Xi)$ of Example 3.1 satisfy Assumption 3: For example, we consider $F(X)\equiv B(t,X(t))$ with $|D_x^\alpha B(t,x)|\le C_\alpha\langle x\rangle^L$ of Example 3.1 (1). Let $T_{k-1}<t\le T_k$. For any multi-index α, there exists a positive constant C_α such that

$$\left|D_{x_k}^\alpha F_{\Delta_{T,0}}(x_k)\right|\le C_\alpha\langle x_k\rangle^L.$$

Set $\chi(\tau)=0$ $(0\le\tau<t)$, $=1$ $(t\le\tau\le T)$ and $\mu((T_{j-1},T_j])\equiv\int_{(T_{j-1},T_j]}d\chi(\tau)$. Then we have $\mu((T_{j-1},T_j])=0$ $(j\ne k)$, $=1$ $(j=k)$. Therefore we can show that $F_{\Delta_{T,0}}$ satisfies (5.17). Next we consider

$$F(X,\Xi)\equiv e^{\int_{[0,T)}B(t,X(t),\Xi(t))dt}$$

with $|D_x^\alpha\partial_\xi^\beta B(t,x,\xi)|\le C_{\alpha,\beta}\lambda(\xi)^{\delta|\alpha|-\rho|\beta|}$ of Example 3.1 (3) with $0=T'<T''=T$. For multi-index e with $|e|=1$, we can write

$$D_{x_s}^e F_{\Delta_{T,0}}=\prod_{j=1}^{J+1}e^{\int_{[T_{j-1},T_j)}B(t,x_j,\xi_{j-1})dt}\times\int_{[T_{s-1},T_s)}(D_x^eB)(t,x_s,\xi_{s-1})dt. \tag{5.18}$$

Set $\mu((T_{s-1},T_s])\equiv\int_{(T_{s-1},T_s]}dt=t_s\le\mu((0,T])\equiv\int_{(0,T]}dt=T<\infty$. Then we can show that $F_{\Delta_{T,0}}$ satisfies (5.17).

Roughly speaking, the measure theory considers the base. However the integration theory considers the area, i.e., the product of the base and the height. We assume (5.17) to the height. (5.17) implies that if the difference of two paths is small, the difference $D_{x_s}F_{\Delta_{T,0}}$ of the two heights can be controlled by $\mu((T_{s-1},T_s])$ with $\sum_{j=1}^{J+1}\mu((T_{j-1},T_j])\le\mu((0,T])<\infty$.

Under this Assumption 3, we can show that $q_{\Delta_{T,0}}(x,\xi_0,x_0)$ of (5.1) converges as $|\Delta_{T,0}|\to0$ as follows.

Lemma 5.4. *Let $0<T\le\mathbf{T}$. For any non-negative integers ℓ_1, ℓ_2, there exist positive constants C_{ℓ_1,ℓ_2}, C'_{ℓ_1,ℓ_2} and a function $q(T,0;x,\xi_0,x_0)$ such that*

$$\left|D_x^\alpha\partial_{\xi_0}^\beta q_{\Delta_{T,0}}(x,\xi_0,x_0)\right|\le AC_{\ell_1,\ell_2}\left(\langle x\rangle+\lambda(\xi_0)+\langle x_0\rangle\right)^L\lambda(\xi_0)^{\delta|\alpha|-\rho|\beta|}, \tag{5.19}$$

and

$$\begin{aligned}&\left|D_x^\alpha\partial_{\xi_0}^\beta\left(q_{\Delta_{T,0}}(x,\xi_0,x_0)-q(T,0;x,\xi_0,x_0)\right)\right|\\&\le AC'_{\ell_1,\ell_2}|\Delta_{T,0}|\left(T+\mu((0,T])\right)\left(\langle x\rangle+\lambda(\xi_0)+\langle x_0\rangle\right)^L\lambda(\xi_0)^{2m+\delta|\alpha|-\rho|\beta|}.\end{aligned} \tag{5.20}$$

for any division $\Delta_{T,0}$ and any multi-indices α, β with $|\alpha|\le\ell_1$ and $|\beta|\le\ell_2$.

At last, we define the class $\mathcal{F}$ of functionals $F(X, \Xi)$ by this Assumption 3.

Definition 1 (Class $\mathcal{F}$ of functionals). Let $F(X, \Xi)$ be a functional of $X_{\Delta_{T,0}}(t)$ and $\Xi_{\Delta_{T,0}}(t)$ in (2.8) and (2.9). We say that $F(X, \Xi) \in \mathcal{F}$ if $F_{\Delta_{T,0}} = F(X_{\Delta_{T,0}}, \Xi_{\Delta_{T,0}})$ satisfies (5.16) and (5.17) of Assumption 3.

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